

STAT 134
Notes on Superposition and Thinning of Poisson Processes
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1 Superposition

Suppose you are an owner of a shop. Women arrive to your shop according to a Poisson process with rate λ (e.g., $\lambda = 4$ women/hour). Similarly, men arrive to your shop according to a Poisson process with rate μ . The two processes are independent.

If we combine the two processes into one (that is, we consider the process of *all* arrivals regardless of gender), then this combined process is also a Poisson process, and its rate is $\lambda + \mu$.

The rate part of this statement is intuitive: if on average 4 women arrive per hour, and 3 men arrive per hour (and the arrivals of men and women are independent), then we have on average 7 customers per hour. The fact that the combined process is a Poisson process is the non-obvious part.

Theorem 1 (Superposition). *If we have two independent Poisson processes with rates λ and μ respectively, then the combined process of the arrivals from both processes is a Poisson process with rate $\lambda + \mu$.*

Proof. To show that the combined process N is a Poisson process with rate $\lambda + \mu$, we need to show that a) the number of total arrivals $N(I)$ in an interval I of length t follows a $\text{Pois}((\lambda + \mu)t)$ distribution, and b) the numbers of total arrivals $N(I_1), \dots, N(I_k)$ in disjoint intervals I_1, \dots, I_k are independent.

For a), simply note that $N(I)$ can be written as $N_1(I) + N_2(I)$, where N_1 and N_2 are the two original Poisson processes; that is, the number of total arrivals in interval I is the sum of the number of arrivals from the first process and the number of arrivals from the second process. We know $N_1(I) \sim \text{Pois}(\lambda t)$ and $N_2(I) \sim \text{Pois}(\mu t)$ and the two are independent. Using an important property of Poisson random variables (see page 226 of the textbook), we have $N(I) \sim \text{Pois}(\lambda t + \mu t) = \text{Pois}((\lambda + \mu)t)$.

For b), note that if I_1, \dots, I_k are disjoint, then $N_1(I_1), \dots, N_1(I_k), N_2(I_1), \dots, N_2(I_k)$ are all independent, so in particular, $N_1(I_1) + N_2(I_1), \dots, N_1(I_k) + N_2(I_k)$ are independent. \square

2 Thinning

To avoid confusion in notation in what follows, we will take $\lambda = p\alpha$ and $\mu = (1 - p)\alpha$ so that the above result states that combining two independent Poisson processes with respective rates $p\alpha$ and $(1 - p)\alpha$ yields a Poisson process with rate α .

Note that in the combined process, we did not keep track of gender; we just had a Poisson process of customer arrivals. If we retain the labeling of the customers as female (Type 1) or male (Type 2), we can say something stronger than the above result. This combined process (with the labels) is the same as if we just started with a Poisson process with rate α , and flipped a coin to choose the label (type/gender) of each arrival independently.

Theorem 2 (Thinning). *The following two labeled Poisson processes have the same distribution.*

(i) *Take two independent Poisson processes with respective rates $p\alpha$ and $(1 - p)\alpha$, and combine them into a single process of arrivals from both processes. Each arrival is labeled Type 1 if it came from the first process, and Type 2 if it came from the second process.*

(ii) *Take a Poisson process with rate α , and for each arrival (independently of other arrivals) flip a p -biased coin and label the arrival as Type 1 (heads) or Type 2 (tails).*

An alternative statement: if we follow procedure (ii) and create one process consisting of only the Type 1 arrivals, and another consisting of only the Type 2 arrivals, then these two resulting processes are independent Poisson processes with respective rates $p\alpha$ and $(1 - p)\alpha$.

The value of the rates is intuitive: if on average you get 8 customers per hour, and for each customer you get to decide what gender they are with a 1/4-biased coin (female if heads, male if tails), then on average you will have 2 female customers per hour and 6 male customers per hour. The non-obvious part is that the female arrivals and male arrivals are each Poisson processes and are independent.

Proof sketch. Consider some interval I of length t . If we follow procedure (i), then the number of Type 1 arrivals $N_1(I)$ and Type 2 arrivals $N_2(I)$ in interval I follow the distributions $\text{Pois}(p\alpha t)$ and $\text{Pois}((1 - p)\alpha t)$ respectively, and moreover they are independent. We will show that if we follow procedure (ii) instead, we still get these two independent Poisson random variables.

Let $N_1(I)$ and $N_2(I)$ be the number of Type 1 arrivals and Type 2 arrivals respectively, under procedure (ii). If $N_1(I) = j$ and $N_2(I) = k$, that means there must have been $j + k$ arrivals of the original α -Poisson process in interval I , and of these $j + k$ coin flips, exactly j of them were heads.

$$\begin{aligned}
 \mathbb{P}(N_1(I) = j, N_2(I) = k) &= \mathbb{P}(j + k \text{ arrivals in } I, \text{ and } j \text{ of these } j + k \text{ coin flips were heads}) \\
 &= \mathbb{P}(j + k \text{ arrivals in } I) \cdot \mathbb{P}(j \text{ heads in } j + k \text{ flips}) \\
 &= e^{-\alpha t} \frac{(\alpha t)^{j+k}}{(j+k)!} \cdot \binom{j+k}{j} p^j (1-p)^k \\
 &= e^{-p\alpha t} \frac{(p\alpha t)^j}{j!} \cdot e^{-(1-p)\alpha t} \frac{((1-p)\alpha t)^k}{k!} \\
 &= \mathbb{P}(N_1(I) = j) \cdot \mathbb{P}(N_2(I) = k).
 \end{aligned}$$

Indeed, under procedure (ii), the joint distribution of $N_1(I)$ and $N_2(I)$ are two independent Poisson random variables with parameters $p\alpha t$ and $(1 - p)\alpha t$ respectively. \square

3 Application

Let us return to our earlier notation: $\lambda = p\alpha$ and $\mu = (1 - p)\alpha$ is equivalent to $\alpha = \lambda + \mu$ and $p = \frac{\lambda}{\lambda + \mu}$. Thus, the above result shows that if we combine two independent Poisson processes with respective rates λ and μ (while retaining labels marking which process each arrival originally came from), then we can instead think of it as a single Poisson process of rate $\lambda + \mu$ with the arrivals labeled by independent flips of a $\frac{\lambda}{\lambda + \mu}$ -biased coin.

For example, let $X \sim \text{Gamma}(5, \lambda)$ and $Y \sim \text{Gamma}(3, \mu)$ be independent, and suppose we want to compute $\mathbb{P}(X < Y)$. Then we can think of X as the time of the 5th arrival in a Poisson process with rate λ , and Y the time of the 3rd arrival in an independent Poisson process with rate μ . If we combine the two processes, we can use it to interpret the event $\{X < Y\}$.

$$\begin{aligned}\{X < Y\} &= \{5\text{th Type 1 arrival is before the 3rd Type 2 arrival}\} \\ &= \{\text{there are at most two Type 2 arrivals before the 5th Type 1 arrival}\}\end{aligned}$$

Using the alternate interpretation of the combined Poisson process, we can just view the labeled arrivals as a sequence of independent $\frac{\lambda}{\lambda + \mu}$ -biased coin flips, and we simply want the probability of ≤ 2 tails before the 5th heads. This probability can be computed using the negative binomial distribution.